UDC 51-72:531

MUHARRAMOV M.A.

DIRECT AND INVERSE PROBLEMS OF PIPELINE CATHODIC PROTECTION

1. Mathematical Model Based on Ordinary Differential Equations

Let a segment of an underground metal pipeline be represented by means of an infinite, semifinite or finite interval of the real line R^1:

$a < x < b, \ -\infty \le a < b \le +\infty$

wherein x denotes a linear co-ordinate on the pipeline. Equality of either a or b to infinity means that for all practical purposes the pipeline segment can be assumed infinite in the corresponding direction. In order to derive an equation governing the distribution of potential on a pipeline, we consider in this paragraph a model of pipeline cathodic protection that is based on an equivalent circuit presented in [1]. Linear resistanse means the resistance of a unit length segment of pipe as measured between its extremities (i.e., junctions to the adjacent segments) and lateral resistance means the resistance to the remote earth of such a segment of buried pipe detached from the rest of the pipeline[1]. Every peace of pipe metal possesses a natural potential, which varies along the pipeline thus causing local micro-currents[2].

Inasmuch as electric currents can be linearly superimposed, we can segregate micro-currents caused by natural potential variations from the currents caused by external sources, studying the latter by means of an ordinary differential equation and relying on field test measurements of natural potentials for the former. If $\rho_1(x), \rho_2(x)$ now denote the linear and lateral specific resistivities of the pipe – i.e., linear and lateral resistivitiesof a unit length pipe segment at a point x – we can write the following differential equation governing the local behaviour of potential shifts on the pipeline:

$$\frac{d^2\phi}{dx^2} - \frac{\rho_1'(x)d\phi}{\rho_1(x)dx} - \frac{\rho_1(x)}{\rho_2(x)}\phi(x) = 0 \quad (1)$$

wherein $\phi(x)$ is the potential shift at a point x. In a particular case when the linear specific resisivity is homogenous throughout the pipeline –i.e., constant- the second term will vanish:

$$\frac{d^2\phi}{dx^2} - \frac{\rho_1}{\rho_2(x)}\phi(x) = 0 \quad (2)$$

Further assuming constancy of the lateral specific resistivity (that is homogeneity of both the coating conductivity and soil conditions) we reduce the equation to the following simplified from, which is widely used in industry:

$$\frac{d^2\phi}{dx^2} - \frac{\rho_1}{\rho_2}\phi(x) = 0 \quad (3)$$

This equation allows for a simple analytical solution: $\phi(x) = \exp\{\pm\alpha x\}$ where $\alpha = \sqrt{\rho_1 / \rho_2}$ is referred to as «attenuation constant»

To make Equations (1)-(3) usable, we must complement them with the following conditions on ϕ:

$$\phi(x_0) = U_0, \ \frac{\phi'(x_0+) - \phi'(x_0-)}{\rho_1(x_0)} =$$

[1] Not the same as «coating resistance» even though lagely determined by it

[2] and corroding sites of relatively positive natural potential

[Muharramov M.A.]

$$= -\int_a^b \frac{\phi(x)}{\rho_2(x)}dx = I, \tag{4}$$

where x_0 is the co-ordinate of the drain point $(a < x_0 < b), U_0$ is the potential shift at the drain point, I is the total current flowing through the pipeline and $\phi'(x_0+)$, $\phi'(x_0-)$ are the right and left-hand derivatives of $\phi(x)$ at the drain point[3].

The integral in the second condition equals the total current flowing through the pipeline, and

$\phi'(x_0+)/\rho_1(x_0), \phi'(x_0-)/\rho_1(x_0)$

are the currents flowing to the right and left of the drain point.

We have effectively reduced our task to solving two Cauchy problems for the ordinary differential equation (1) on the intervals (a,U_0) and (U_0,b) subject to Condition (4). In case the drain point is at one of the segment's extremities, we will get:

$$\phi(a) = U_0, \phi'(a)/\rho_1(a) = I \tag{5}$$

for the left-hand drain point, and

$$\phi(b) = U_0, \phi'(b)/\rho_1(b) = I \tag{6}$$

otherwise. Combining (5) or (6) with (3) we obtain a simplified model of potential distribution currently utilised in the pipeline and power transmission industries (e.g., Aramco Design Practice).Problem (2,4) can be used for restoration of inhomogeneous coating conductivity on the basis of test measurements –i.e., solving the inverse problem of cathodic protection. For this purpose we assume that the function $\rho_2(x)$ is partially constant –i.e., takes discrete values on sub-intervals that form a partition of the interval (a,b):

$$a = \xi_0 < \xi_1 < \xi_2 < \ldots < \xi_{N-1} < \xi_N = b,$$
$$\rho_2(x) = S_i, if\ \xi_{i-1} < x < \xi_i,$$
$$i = 1,2,\ldots,N \tag{7}$$

This condition means that coating quality is the same throughout any pipe segment (ξ_{i-1},ξ_i) $i = 1,2,\ldots,N$, but may differ among different segments (e.g., «new« and «old» pipe). Solution of the problem (2,4,7) therefore comes down to solving a series of problems (3,5) (to the right of the drain point) and (3,6) (to the left thereof) with the following reconciliation conditions:

$$\phi'(\xi_i-) = \phi'(\xi_i+), \xi_i \neq x_0, \phi(\xi_i-) = = \phi(\xi_i+), i = 1,2,\ldots,N. \tag{8}$$

Assuming that $\xi_{j-1} < x_0 < \xi_j$, $1 \le j \le N$ and $\phi(\xi_1), \phi(\xi_2),\ldots,\phi(\xi_N), \phi(x_0), I$ are known,we can use the following algorithm for calculating the specific resistivities $S_i, i = 1,2,\ldots,N$:

1. Solve the following systems of non-linear equations with respect to two unknowns $\phi'(x_0+)$ and S_j:

$$\begin{cases} \phi(\xi_j) = \phi(x_0)\cosh\left(\sqrt{\frac{\rho_1}{S_j}}(\xi_j - x_0)\right) + \\ + \phi'(x_0+)\sinh\left(\sqrt{\frac{\rho_1}{S_j}}(\xi_j - x_0)\right) \\ \phi(\xi_{j-1}) = \phi(x_0)\cosh\left(\sqrt{\frac{\rho_1}{S_j}}(x_0 - \xi_{j-1})\right) + \\ + (\rho_1 I - \phi'(x_0+))\sinh\left(\sqrt{\frac{\rho_1}{S_j}}(x_0 - \xi_{j-1})\right) \end{cases} \tag{9}$$

2.Calculate

$$\phi'(\xi_j) = \phi(x_0)\sqrt{\frac{\rho_1}{S_j}}\sinh\left(\sqrt{\frac{\rho_1}{S_j}}(\xi_j - x_0)\right) + + \phi'(x_0+)\sqrt{\frac{\rho_1}{S_j}}\cosh\left(\sqrt{\frac{\rho_1}{S_j}}(\xi_j - x_0)\right),$$

[3] Note that the function $\phi(x)$ is not differentiable at the drain point unless the latter is at one of the segment's extremities - the abso-lute value of the potential shift has a «peak» at the drain point, sloping down in both directions

$$\phi'(\xi_{j-1}) = -\phi(x_0)\sqrt{\frac{\rho_1}{S_j}}\sinh\left(\sqrt{\frac{\rho_1}{S_j}}(x_0 - \xi_{j-1})\right) - (\rho_1 I - \phi'(x_0+))\sqrt{\frac{\rho_1}{S_j}}\cosh\left(\sqrt{\frac{\rho_1}{S_j}}(x_0 - \xi_{j-1})\right)$$

3. For each $i = j+1,\ldots,N$ perform the following procedure:
Solve the following equation with respect to S_i:

$$\phi(\xi_i) = \phi(\xi_{i-1})\cosh\left(\sqrt{\frac{\rho_1}{S_i}}(\xi_i - \xi_{i-1})\right) + \phi'(\xi_{i-1})\sinh\left(\sqrt{\frac{\rho_1}{S_i}}(\xi_i - \xi_{i-1})\right)$$

Calculate:

$$\phi'(\xi_i) = \phi(\xi_{i-1})\sqrt{\frac{\rho_1}{S_i}}\sinh\left(\sqrt{\frac{\rho_1}{S_i}}(\xi_i - \xi_{i-1})\right) + \phi'(\xi_{i-1})\sqrt{\frac{\rho_1}{S_i}}\cosh\left(\sqrt{\frac{\rho_1}{S_i}}(\xi_i - \xi_{i-1})\right)$$

4. For each $i = j-1,\ldots,2,1$ perform the following procedure:
Solve the following equation with respect to S_i:

$$\phi(\xi_{i-1}) = \phi(\xi_i)\cosh\left(\sqrt{\frac{\rho_1}{S_i}}(\xi_i - \xi_{i-1})\right) + \phi'(\xi_i)\sinh\left(\sqrt{\frac{\rho_1}{S_i}}(\xi_i - \xi_{i-1})\right)$$

Calculate:

$$\phi'(\xi_{i-1}) = -\phi(\xi_i)\sqrt{\frac{\rho_1}{S_i}}\sinh\left(\sqrt{\frac{\rho_1}{S_i}}(\xi_i - \xi_{i-1})\right) - \phi'(\xi_i)\sqrt{\frac{\rho_1}{S_i}}\cosh\left(\sqrt{\frac{\rho_1}{S_i}}(\xi_i - \xi_{i-1})\right)$$

System (9) can be solved by means of Newton's linearization method, but requires sufficiently accurate initial approximations of $\phi'(x_0+)$ and S_j. In the event the lateral specific resistivity is homogeneous throughout the pipeline, the above algorithm can determine the value of the lateral resistivity on the basis of test measurement at three points: two test points at the extremities of the pipeline segment under consideration and a drain point in between them. In such a case, the algorithm will terminate after Step 1-i.e., after the calculation of the lateral specific resistivity S_1 on the segment $(\xi_0, \xi_1) = (a,b)$.

Furthermore, if the lateral specific resistivity is constant throughout the pipeline, its value can be restored from a single drain-point measurement using Ohm;s law. To accomplish this, we nedd to find out how pipeline resistance can be expressed via its length and lateral resistivity.

Assume that x is the length of a buried pipe,0 corresponds to the pipe's left end and $R(x)$ denotes the resistance between the point 0 and the remote earth. The resistance as a function of the length satisfies the following non-linear ordinary differential equation:

$$\frac{dR(x)}{dx} = -\frac{R^2(x)}{\rho_2(x)} + \rho_1(x) \qquad (10)$$

with the following initial-value (Cauchy) condition:

$$R(0) = +\infty \qquad (11)$$

Note that Equation (10) allows for variable linear and lateral specific resistivities.

If the pipe's left end is attached to a metal structure that has a finite resistance R_0 to earth (e.g., terminal piping connected to a terminal earthing system), the condition (11) should be replaced with the following one:

$$R(0) = R_0 \qquad (12)$$

We can now calculate the resistance of pipe-to-earth system between a point x_0 on a pipeline of length l and the remote earth in case of contant resistivities:

$$R_{total} = 1 \; / \; R^{-1}(R_{left}, x_0) + + R^{-1}(R_{right}, l - x_0) \quad (13)$$

wherein R_{left} and R_{right} are «terminal» resistances of the pipeline and $R(a,x)$ denotes the solution to (10) with $\rho_1(x) \equiv \rho_1 = const$, $\rho_2(x) \equiv \rho_2 = const$, satisfying the initial condition $R(0) = a$ – i.e., .

$$R(a,x) = \begin{cases} \sqrt{\rho_1/\rho_2} + \frac{\rho_1}{a} \times \\ \times \tanh\left(x\sqrt{\rho_1/\rho_2}\right) / \\ / \; \frac{1}{a}\sqrt{\rho_1/\rho_2} + \frac{1}{\rho_2} \times \\ \times \tanh\left(x\sqrt{\rho_1/\rho_2}\right) \\ \quad if \quad a \neq \infty \\ \sqrt{\rho_1/\rho_2} / \tanh\left(x\sqrt{\rho_1/\rho_2}\right) \\ \quad if \quad a = \infty \end{cases} \quad (14)$$

Now if the current thruogh the pipeline and the potential shift at the drain point (hence R_{total}) are known, we can resolve Equation (13) with respect to ρ_2 and so obtain the value of (homogeneous) lateral resistivity. Albeit this approach depends on the constancy of the lateral resistivity, it can be used for obtaining initial approximations of S_j and $\phi'(x_0+)$ in System (9):

$$S_j = \rho_2, \; \phi'(x_0+) = = -\frac{\phi(x_0)\rho_1}{R(R_{right}, l - x_0)} \quad (15)$$

where ρ_2 is found from Equation (13) with $R_{total} = -\frac{I}{\phi(x_0)}$ -note that $\phi(x_0)$ is the potential at the drain point measured with respect to the remote earth less the natural potential.

This technique of back-calculating (in-) homogeneous lateral resistivity has been implemented in CP DESIGN cathodic protection design software (see [2]). Inasmuch as the coating specific resistivity is the deciding factor in the magnitude of the lateral resistivity, the latter can be utilised to estimate the value of the former, soil conditions being neglected. The coating resistivity can be obtained from the lateral specific resistivity as follows:

$$\rho_{coat}(x) = \pi D(x)\rho_2(x) \quad (16)$$

wherein $D(x)$ is the outer pipeline diameter at a point x. The linear specific resistivity is likewise expressed through pipe metal (steel) resistivity and pipe dimensions:

$$\rho_{metal}(x) = \rho_1(x)(W(x)\pi D(x) - - W^2(x)) \quad (17)$$

W is the wall thickness.

2. –Mathematical Model Based on Differential Equations with Partial Derivatives

In this paragraph, we will spell out the equations governing potential distribution in the ground as well as throughout the pipeline, and get rid of the previous assumption that potentials have to be measured with respect to a remote earth. A finite segment of buried pipeline can be represented as a cylinder in the three-dimensional Euclidean Space:

$$P = [-a, +a] \times \{(y,z) : y^2 + z^2 \leq r^2\} \quad (18)$$

wherein r is the pipeline radius, and the segment length equals $2a$. If we denote potential in the ground through u and the pipeline potential change via ψ, the system goverinig u and ψ can be written as follows:

$$div(\sigma(x,y,z)\nabla u(x,y,z)) = 0, \\ u \in W_2^1((-\infty,+\infty) \times (-\infty,+\infty) \times \times (-\infty, h) \setminus P), \; h > 0 \quad (19)$$

$$\frac{d^2\psi(x,\phi)}{dx^2}+\frac{\left(\frac{d\rho_1(x)}{dx}\right)}{\rho_1(x)}\frac{d\psi(x,\phi)}{dx}- \qquad (20)$$
$$-\frac{\rho_1(x)}{\rho_2(x)}\psi(x,\phi)=0$$
$$-a\le x\le a,\ 0\le\phi\le 2\pi,$$
$$\psi\in W_2^1([-a,a]\times[0,2\pi])$$
coupled with the following conditions:
$$\psi(0,\phi)+\psi_{nat}(0,\phi)-u(x_0,y_0,z_0)=U_0,$$

$$\frac{d\psi(0-,\phi)}{dx}-\frac{d\psi(0+,\phi)}{dx}=\rho_1(0)I,$$
$$\left.\frac{\partial u}{\partial n}\right|_{(x,y)\in R^2,z=h}=0,\ u(\infty)=0, \qquad (21)$$
$$\int_{\partial P}\frac{(\psi(x,\phi)+\psi_{nat}(x,\phi)-u(x,y,z))}{\rho_2(x)}dS=$$
$$=-\int_{\partial P}\frac{\langle\nabla u(x,y,z)\bullet \mathbf{dS}\rangle}{\rho_2(x)}=I \qquad (22)$$
In these equations, $z=h$ represents the ground surface, (x,ϕ) are co –ordinates on the cylinder P(i.e., pipeline), $\sigma(x,y,z)$ is soil resistivity, U_0 is the drain point potential change, $\psi(x,\phi)$ and $\psi_{nat}(x,\phi)$ are the pipeline potential change and natural potential, respectively. The other variables and functions are either self-explanatory or have been described in the Paragraph *I*. The drain point in the above equations is assumed to be at the point $x=0, y=0, z=0$, but any other drain point can be written similar equations for. Incidentally, the drain point potential change is shown as specified with respect to a point (x_0,y_0,z_0) in the ground rather than a «remote earth». The notation W_2^1 denotes Sobolev Space of functions that have the frist order partial derivatives in L_2.

The system (19)-(22) lends itself to a numerical solution, allows for complicated soil conditions and test measurements between the pipeline and any points in the ground.

References

[1]. Peabody A.W. *Control of Pipeline Corrosion.* NACE, Houston, Texas, 1970.
[2]. Lax K.C., Muharramov M.A. CP DESIGN: *Cathodic Protection Design System.* Corrocont UK, 1997.

Məhərrəmov M.A.

NƏQLİYYAT BORULARI KATOD MÜHAFİZƏSİNİN DÜZ VƏ TƏRS MƏSƏLƏLƏRİ

İşdə, adi və xüsusi törəməli diferensial tənliklərdən istifadə etməklə magistral boru kəmərlərinin katod müdafiə sistemi üçün iki riyazi model təklif edilmişdir. Boru kəməri parametrlərinin, potensialın ölçülməsi nəticələrinə əsasən, tapılması kimi tərs məsələnin həll üsulu və uyğun alqoritm verilmişdir. Bu işdəki metod müəllif tərəfindən işlənmiş və sənayedə öz tətbiqini tapmış proqram təminatının əsasını təşkil edir.