Efficient Finite-difference Modelling of Acoustic Wave Propagation in Anisotropic Media with Pseudo-sources

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SUMMARY

This work proposes a technique for deriving fast finite-difference algorithms for numerical modelling of acoustic wave propagation in anisotropic media. The technique is deployed in the case of transversely isotropic media to implement a kinematically accurate shear-free fast finite-difference modelling method. This results in a significant reduction of pseudo-shear artifacts compared to similar kinematically accurate finite-difference methods.
Introduction

Transverse isotropy and orthorhombic media are of significant interest for imaging applications (Grechka, 2009). The “pseudo-acoustic” wave-propagation modeling method of Alkhalifah (1998) is the anisotropic counterpart of isotropic acoustic modeling. The existing computationally cheap anisotropic finite-difference methods not based on singular pseudo-differential operators either suffer from “pseudo-shear” artifacts or employ approximations that break down for strong anisotropy (Fowler et al., 2010; Zhan et al., 2012). An alternative approach based on solving pseudo-differential evolutionary equations (Etgen and Brandsberg-Dahl, 2009), while free from pseudo-shear artifacts, requires multiple Fourier transforms at each time step and becomes computationally expensive for complex anisotropies. Another alternative based on solving nonlinear equations (Xu and Zhou, 2014) produces kinematically accurate results but the nonlinearity may pose challenges in practical implementation, especially in derivative-based inversion methods. In this work we propose a computationally efficient finite-difference wave propagation modeling method for anisotropic media that is largely free of pseudo-shear artifacts. We achieve this by projecting seismic sources onto the shear-free solution space of a pseudo-differential operator (Maharramov, 2014) while using conventional coupled equations for propagation.

Figure 1

Left) Test model with smooth and sharp VP gradients and constant $\varepsilon = 0.3$ and $\delta = 0.1$. Right) Test model with two anisotropic inclusions; $\varepsilon = 0.3, \delta = 0.1$ in region I, $\varepsilon = 0.4, \delta = 0.05$ in region II, $\varepsilon = 0.5, \delta = 0.15$ in region III, color scale is for VP.

Method

We start with the equation for $V(\theta)$ in a VTI medium (Tsvankin, 1996)

$$\frac{V^2(\theta)}{V_P^2} = 1 + \varepsilon \sin^2 \theta - \frac{1}{2} \pm \frac{1}{2} \sqrt{(1 + 2\varepsilon \sin^2 \theta)^2 - 2(\varepsilon - \delta) \sin^2 2\theta}, \quad \sin \theta = V(\theta) \frac{\partial}{\partial x} \frac{\partial}{\partial t}, \quad (1)$$

where $V(\theta)$ is velocity of acoustic wave propagation in the direction of angle $\theta$ from vertical, $V_P$ is the vertical pressure wave velocity, the vertical shear-wave velocity $V_S$ is assumed zero, and $\varepsilon$ and $\delta$ are the Thomsen parameters (Thomsen, 1986). Note that here and in the subsequent analysis we consider two-dimensional VTI, however, the results naturally extend to three dimensions by identifying $k_x$ with the radial wavenumber—see, e.g., (Maharramov and Nolte, 2011). We use the equivalence $k_u = -i \partial / \partial u$ in (1), where $u$ is an arbitrary variable, to stress that the phase velocity equation can be interpreted as both a dispersion relation and a pseudo-differential operator. We extract the branch of the square root with the positive sign in (1), corresponding to the (higher) acoustic wave velocity. The resulting dispersion relation can be interpreted as an evolutionary pseudo-differential operator governing kinematically accurate propagation of the pressure wave:

$$\frac{\partial^2}{\partial t^2} - V_P^2 \left( \frac{2}{z_x} \right) \frac{\Delta}{2} - \varepsilon \left( \frac{2}{z_x} \right) V_P^2 \left( \frac{2}{z_x} \right) \frac{\partial^2}{\partial x^2} =$$

$$V_P^2 \left( \frac{2}{z_x} \right) \frac{\Delta}{2} \left[ 1 + 2\varepsilon \left( \frac{2}{z_x} \right) \frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial z^2} - 8 \left( \varepsilon \left( \frac{2}{z_x} \right) - \varepsilon \left( \frac{2}{z_x} \right) \right) \frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial z^2} \frac{1}{\Delta^2}, \quad (2)$$
where Δ is the Laplace operator in x, z, and “2” over x and z means that the multiplication by functions of spatial variables follows the application of differential operators in the pseudo-differential operator sense (Maslov, 1979). Operator (2) can be efficiently applied numerically to a function with a spatially bounded support such as seismic source and receiver data, as is the case in our method. An alternative to solving the full pseudo-differential operator equation (2) is to use approximations of the acoustic-velocity branch in (1) (Maharramov, 2014), leading to various existing spectral modeling techniques.

Figure 2 Left) Shear artifacts in the solution of (3) for the gradient model of Figure 1, with sources injected in the r-component. Right) Pseudo-shear artifacts in the solution of (3) with sources injected in the q-component are significantly weakened but still present.

The Finite-Difference Method

We square the pseudo-differential operator equation (2) so as to get rid of the square root, and obtain the following system of coupled second-order partial differential equations (Alkhalifah, 2000):

\[
\frac{\partial^2 q}{\partial t^2} = V_{P_{\text{Hor}}} \frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial z^2} + V_P^2 \frac{\partial^2 q}{\partial P_{\text{PNMO}}} \frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial t^2} = q, \tag{3}
\]

where \( r(z,x,t) \) and \( q(z,x,t) \) are the pressure field and its second temporal derivative, and

\[
V_{P_{\text{Hor}}}(z,x) = V_P(z,x) \sqrt{1 + 2 \epsilon(z,x)}, \quad V_{P_{\text{PNMO}}}(z,x) = V_P(z,x) \sqrt{1 + 2 \delta(z,x)}.
\]

Since the resulting system now includes the branch with the negative square root in (1), solution of this system may suffer from pseudo-shear artifacts as shown in Figure 2. The artifacts can be reduced by injecting sources into the second component \( q \) (Fowler et al., 2010); however, they are still present—see Figure 2 Right). The pseudo-differential operator equation (2) can be used to reduce the unwanted artifacts (appearing as the “diamond”-shaped inverted wavefront in the figure). Equation (1) and the corresponding pseudo-differential equation do not describe any pressure to shear conversion but rather govern the independent propagation of pressure and shear waves. The same is true of the “coupled” system of differential equations. Consequently, any pseudo-shear artifacts that appear in a solution to the coupled system of differential equations is likely due to the pseudo-shear modes present in the wave field. We can use the fact that the system of two coupled equations requires injecting two sources, to create a “pseudo-source” to be injected into one of the components so as to suppress the shear modes. More specifically, if \( \phi(z,x,t) \) is a time-dependent source function, then at each time step component \( r \) is injected with \( \phi \), and component \( q \) is injected with the result of applying the spatial part of the pseudo-differential operator (2) to \( \phi(z,x,t) \):

\[
\begin{align*}
\Delta r(z,x,t_n) & = r(z,x,t_n) + \phi(z,x,t_n), \\
\Delta q(z,x,t_n) & = q(z,x,t_n) + V_P^2 \left( \frac{\partial \phi}{\partial t} \right) \left\{ \frac{\Delta}{2} + \epsilon (\zeta,z) \frac{\partial^2}{\partial x^2} + \\
& + \frac{\Delta}{2} \sqrt{\left[ 1 + 2 \epsilon (\zeta,z) \frac{\partial^2 \left( z^2 \frac{\partial \phi}{\partial x^2} \right)}{\partial x^2} \right]^2} - 8 \left( \epsilon (\zeta,z) - \delta (\zeta,z) \right) \frac{\partial^2 \left( z^2 \frac{\partial \phi}{\partial x^2} \right)}{\partial x^2} \frac{\partial^2 \left( z^2 \frac{\partial \phi}{\partial x^2} \right)}{\partial z^2 \partial \Delta^2} \right\} \phi,
\end{align*}
\]
followed by a finite-difference time propagation step of system (3). This procedure ensures that the two-component source in the right-hand side of (4) satisfies equation (2). Since solutions of (2) are shear-free, the injected sources will not give rise to shear modes because the solution of (3) is effectively projected onto the solution space of (2). One important consequence of this projection is the resulting stability of system (3) despite our assumption of zero shear-wave velocity (Fowler et al., 2010). Finally, a popular technique of reducing shear artifacts is to introduce an isotropic layer around sources and receivers. In that case accurate shear-free projection still requires computation of both pseudo-source components in equations (4), but the operator for $q$ becomes a scaled Laplacian.

**Example**

We test our method by modeling wave propagation from a Ricker source through the heterogeneous gradient model of Figure 1 Left) and inclusion model of Figure 1 Right). Figure 3 Right) features the result of applying our pseudo-source finite-difference method to the gradient model. The corresponding result obtained by solving the full pseudo-differential operator equation (2) is shown in Figure 3 Left). Results for the inclusion model are shown in Figure 4. Note the apparent absence of pseudo-shear artifacts in all cases. Although we use the full pseudo-differential operator for generating the pseudo-source in (4), the fact that the source is localized makes this computationally efficient.

![Figure 3](image)

**Figure 3** Left) Solution of the full pseudo-differential operator equation (2) for the gradient model. Right) Solution of (3) for the gradient model with pseudo-sources. Note the good agreement with the result of solving the full pseudo-differential operator equation in the left panel.

While adding the pseudo-source (4) ensures that the solution of the coupled system (3) stays within the space of solutions of (2) in the continuous limit $\Delta t \to 0$, sharp contrasts in $\varepsilon$ and $\delta$ may introduce numerical approximation errors resulting in non-negligible pseudo-shear modes. Indeed, applying the method to the inclusion model of Figure 1 Right), we can detect weak high-wavenumber artifacts (marked “A”) within the zoom-in plots of an elliptical inclusion in Figure 5 Right). However, such high-wavenumber numerical artifacts can be easily filtered out, further weakened by smoothing $\varepsilon, \delta$ models, or by ensuring that the discretization of operator (2) is a more accurate approximation to the acoustic branch of the discretized coupled system (3) (Maharramov, 2014).

**Conclusions**

The proposed pseudo-source finite-difference method allows us to take advantage of the computationally cheap finite-difference solvers while achieving a significant reduction of pseudo-shear artifacts. The method is kinematically accurate for VTI media, and can be extended in principle to other kinds of anisotropy. While our implementation is based on using the coupled system (3) of Alkhalifah (2000), the method can be adapted to use equivalent systems (Fowler et al., 2010). In that case the two-component source becomes a linear combination of the true source and pseudo-source terms, with the coefficients of the linear combination determined by the relationship between solutions of the two equivalent systems.
Figure 4 Left) Solution of the full pseudo-differential operator equation (2) for the inclusion model. Right) Solution of (3) for the inclusion model of Figure 1 with pseudo-source injection (4). Note the apparent absence of pseudo-shear artifacts in the finite-difference solution.

Figure 5 Zoom-in of Figure 4 solutions for Left) the pseudo-differential operator (2) and Right) system (3) with pseudo-source injection (4). Note the weak high-wavenumber artifacts like the one marked “A” in the right panel.

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References